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On Minimum Distance Problem

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Abstract

This study provides a clear-cut solution to a minimum distance problem, in particular, the problem of finding the minimum distance from a point to a line to another point on the same side of the line. The straightforward solution is a Pythagorean relation or formula which can be derived through geometrical construction and reasoning, and analytical approach using differentiation, particularly, the application of extreme-value theorem. Such formula is vital in solving minimum distance problems with greater ease, accuracy and speed. This will lessen the cost and waste of materials in practical engineering and business applications.

Keywords: minimum distance formula; geometrical construction; extreme-value theorem; optimization

1. Introduction

Mathematicians, engineers, economists and business professionals have the same opinion that to remain competitive in today's global economy, businesses need to minimize cost, waste and production time while maximizing productivity, profit and return of investment. With this backdrop, the researchers agree to the statement of Smith and Minton [1] that it is increasingly important to use mathematical methods to maximize and minimize various quantities of interest.

Mathematics is the study of relationships among quantities, magnitudes, and properties and of logical operations by which unknown quantities, magnitudes, and properties may be deduced. Berggren [2] emphasized that mathematics, in the past, was regarded as the science of quantity, whether of magnitudes as in geometry, or of numbers as in arithmetic, or the generalization of these two fields as in algebra.

A well-known and a very important branch of mathematics is the calculus. For Seeley [3], calculus has been the principal language of science ever since the seventeenth century, when it was invented, and it is likely to continue in this central role for some time to come. Kalmanson and Kenschaft [4] said that calculus is one of the most beautiful subjects ever created by the human mind. They defined calculus as the study of change, both instantaneous change as in differential calculus and total change as in integral calculus; and change is such a pervasive part of modern life that the subject became useful and essential in the study of many diverse fields.

Optimization problems usually arise in business and industry today. It is in this position that the power of calculus can be brought into, i.e. solving problems involving finding a maximum or a minimum. According to Dick and Patron [5], the full power of calculus can be unleashed if one can model a problem situation and its constraints as functional expressions or as equations or inequalities. Then, a problem solving process can be applied, for instance, the four basic steps in mathematical problem solving provided by Polya (1887-1985) as follows: understand the problem, devise a plan, carry out the plan, and look back.

Several methods can be applied in problem solving; some are long and cumbersome while others are short and easy. Smith and Minton [1] suggested that mathematicians should always seek the shortest and most efficient computations in problem solving but should also remember that shortcuts must always be carefully proved. Mame [6], referring to the computation of the derivative of algebraic function, also said that lengthy and laborious computations could possibly lead to errors in simplifying solutions. In his Masters thesis, he developed corollaries of some basic theorems in differential calculus for the purpose of simplifying the application of the theorems and minimizing steps in computations, thereby saving time used in computing.

One problem that mathematicians need to solve in the shortest and fastest possible way is the minimum distance problem. This problem can be found in both geometry and calculus books. For instance, on page 231 of their geometry book [7], Moise and Dawas Jr. have this problem: *Given a line L and two points, P and Q, on the same side of L. Find the point R on L for which PR + RQ is as small as possible.* This is shown in Figure 1.

Figure 1: The Minimum Distance Problem

Schwartz had an analogous problem on page 427 of his calculus book [8] as follows: *Towns A and B, situated inland on the same side of a straight river, want to erect a jointly owned pumping station P on the river. Where should P be located if the length of pipe to be used from P to A and from P to B is to be minimum?* Leithold, on page 310 of his book [9], gave specific distances between the towns and from the river to each town in a similar problem of finding *where the pumping station should be located so that the least amount of piping is required: Two towns A and B are to get their water supply from the same pumping station to be located on the bank of a* *straight river that is 15 km from town A and 10 km from town B. The points on the river nearest to A and B are 20 km apart and A and B are on the same side of the river.*

On the same book, Leithold discussed that such problems can be solved by the application of the *extreme-value theorem*, the proof of which, he said can be found in advanced calculus text. The theorem is stated as follows: *If the function f is continuous on the closed interval* [a, b], *then f has an absolute maximum value and an absolute minimum value on* [a, b]. He further explained that an absolute extremum of a function continuous on a closed interval must be either a relative extremum or a function value at an endpoint of the interval.

Such solution provided by Leithold and other calculus textbook authors entails finding the derivative of a function and employs somewhat lengthy computations. Hence, in order to provide a straightforward solution, this study was conducted. The aims of the present study are two-fold: (1) Derive the formula for solving the minimum distance problem (a) geometrically and (b) analytically; and (2) Present problems and solutions to illustrate the use of the derived formula.

The whole point of deriving a formula is that the problem solver can go through a line of reasoning only once, and then apply the results whenever there is a need for them, instead of repeating the same reasoning process over and over again. It is considered by many that the most vital tool in solving a mathematical problem is a formula.

2. Methodology

This study is a basic research that employs mathematical exploration and experimentation. The researchers explored the varied concepts regarding the subject of study presented in different areas of mathematics namely, plane geometry, analytic geometry and differential calculus. In this study, the derivations of the formula for calculating the minimum distance were done in two ways: the geometrical approach and the analytical approach. A series of mathematical experimentations and computations was also done in order to establish the proofs of arguments and the use of the derived formula for different cases.

Results of the study were also compared with previously established principles. Aside from the Pythagorean relation and the concepts of minimum distance in geometry and calculus, one principle that is found to be very much related is the Fermat's principle of optics, which states that *the light traveling from one point to another follows the path for which the total travel time is minimum*. In a uniform medium, the paths of *minimum time* and *shortest distance* turn out to be the same, so that the light, if unobstructed, travels along a straight line, as mentioned in [10].

3. Results and Discussion

The formula for finding the minimum distance from a point to a line to another point on the same side of the line is given by

$$d = \sqrt{(a+b)^2 + c^2} ,$$

where *d* is the minimum distance, a is the distance from the first point to the line, b is the distance from the second point to the line and c is the distance between the two points in the line nearest the two given points (see Figure 2).

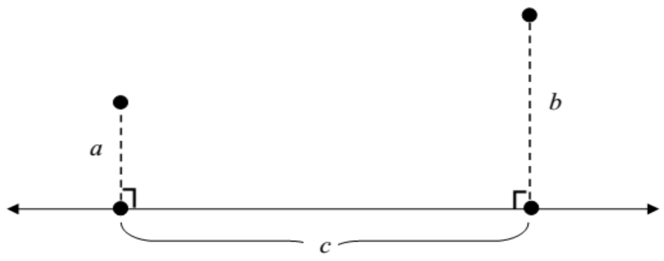

Figure 2: Distances a , b and c of the Minimum Distance Formula

In addition, the formula for locating the point in the line that will lead to having the minimum distance is given

$$x = \frac{ac}{a+b},$$

where *x* is the distance from the point in the line nearest the first point to the point in the line where the two given points are to be connected to give the minimum distance and *a*, *b* and *c* are the distances described above.

3.1. Derivation of the Formula by Geometrical Approach

The minimum distance formula given above can be derived using geometrical reasoning as follows:

Given a plane, a line L and two points A and B on the plane lying on the same side of L, let *a* be the distance from A to L, *b* be the distance from B to L, A″ be the point in L such that $d(A, A'') = a$, B″ be the point in L such that $d(B, B'') = b$, and *c* be the distance from A″ to B″ or symbolically, $d(A'', B'') = c$ (see Figure 3).

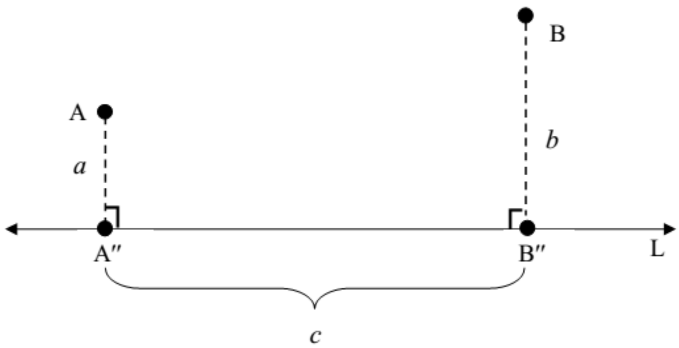

Figure 3: Minimum Distance Problem in Geometry

Then, let P be a point in L such that the distance from A to P to B is to be the minimum distance possible denoted by *d*. Symbolically, *d = d(A,P) + d(P,B).*

Next, constructing a point B' to be the reflection of B with respect to L produces *d(B,B")= d(B",B') = b*. Then, connecting A and B' shows a line segment intersecting L at a point. It is very clear that the minimum distance between A and B' is equal to the length of the segment connecting the two points such that these points are the endpoints of the segment since the shortest distance between two points is through the straight line connecting them (see Figure 4).

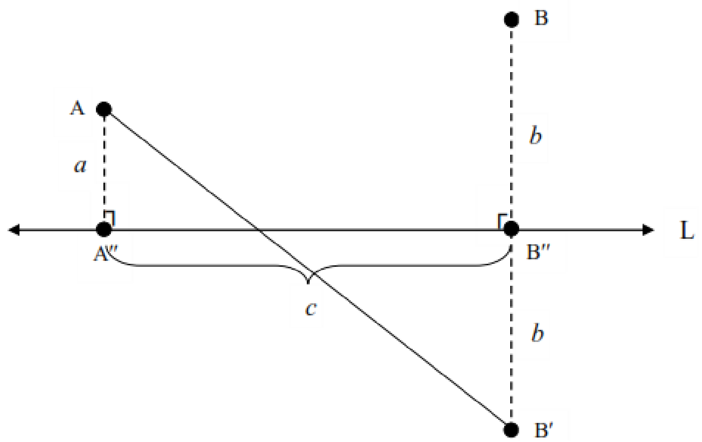

Figure 4: Distance between A and B′

Now, let R be the point of intersection of L and segment AB'. Since distance is preserved under reflection across the line, *d(R,B') = d(R,B)* and hence, *d(A,R)+d(R,B') = d(A,R)+d(R,B).* Thus, the point P in L such that the distance from A to P to B is to be the minimum distance possible is that same point R (see Figure 5).

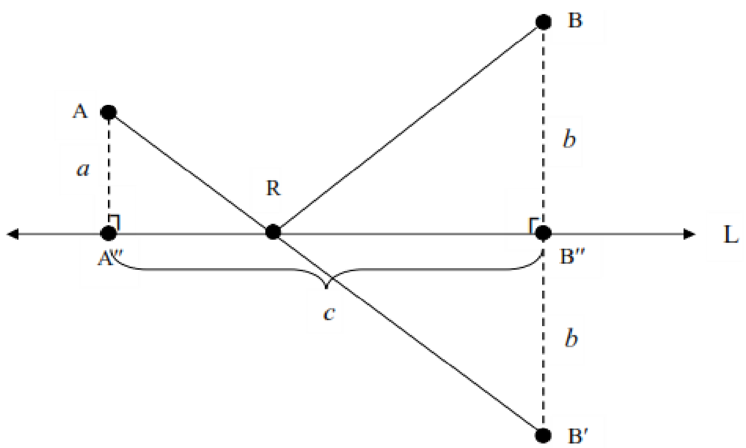

Figure 5: Distance from A to L to B is the same as Distance from A to B′

Since R and P refer to the same point, *d(A,R)+d(R,B')=d(A,P)+d(P,B').* Next, by constructing another point B‴, *c* units away from B′ to the left, it can form a right triangle with vertices A, B′ and B‴ (see Figure 6).

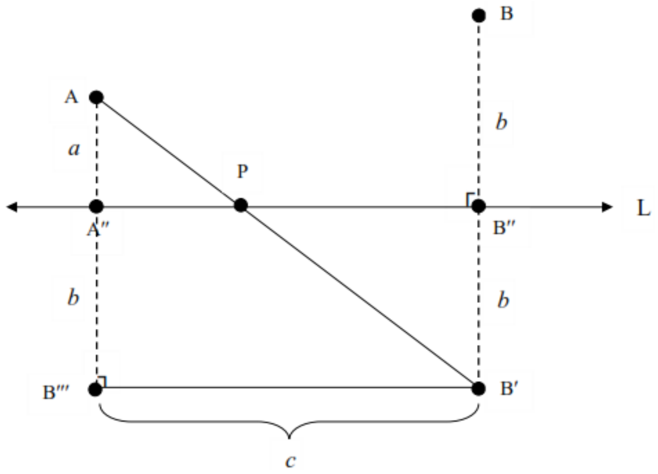

Figure 6: The Right Triangle AB′B‴

It can now be easily seen that the Pythagorean relation holds and hence,

$$d = d(A, B') = \sqrt{(a+b)^2 + c^2}\ .$$

In addition, the location of P can also be determined specifically by using the formula

$$x = \frac{ac}{a+b}$$

Derived as follows:

Looking at Figure 7, since A, P and B' are in a line, $\angle$APA" and $\angle$B'PB'' are vertical angles. Then, let *x* be the distance from A" to P and hence, *c–x* is the distance from P to B".

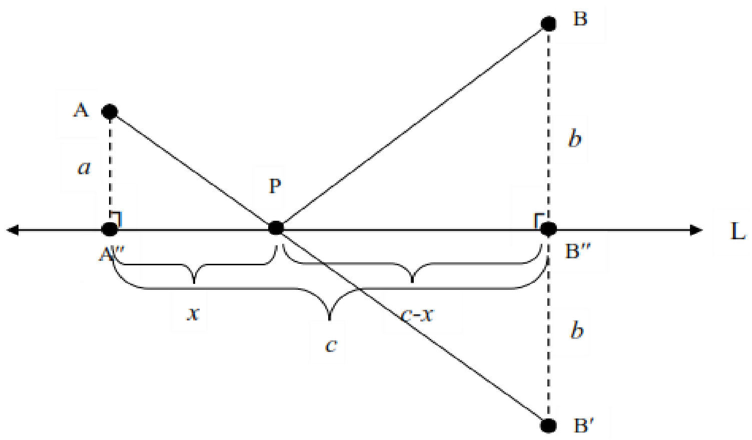

Figure 7: The Location of P

Then, tan (*APA”)* = $\frac{a}{x}$ and tan (*B’PB”)* = $\frac{b}{c-x}$, and since the measure of ∠ APA” is equal to that of ∠ B’PB”, tan (APA”) = tan (B’PB”). Hence, $\frac{a}{x} = \frac{b}{c-x}$ and solving gives $x = \frac{ac}{a+b}$, which is used to locate P in L from A”.

3.2. Derivation of the Formula by Analytical Approach

Given two points A and B on the same side of line L, derive the formula for the minimum distance from A to L to B using differentiation, in particular, applying the extreme-value theorem. Refer to Figure 8 for the following discussion:

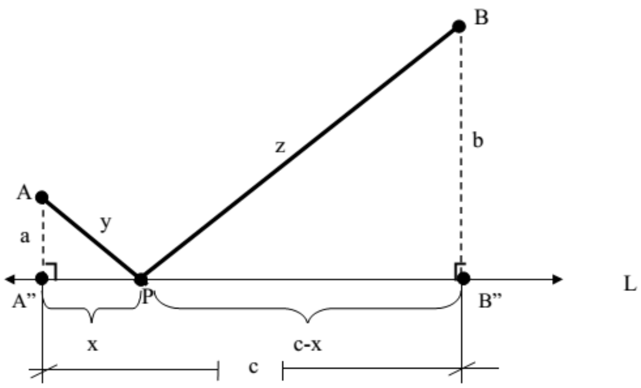

Figure 8: Minimum Distance Problem in Calculus

To begin with, let *a* be the distance from A to L and *b* be the distance from B to L. Then, let A” be the point in L such that *d (A, A”) = a;* B” be the point in L such that *d (B, B”) = b;* and *d (A”, B”) = c*. Further, let P be a point in L and let the distance from A to P to B be minimized. Also, let *d (A, P) = y* , *d (P, B) = z* and *d (A”, P) = x*.

Then, let $d = y + z$ be the distance from A to P in L to B that needs to be minimized. This is clearly an application of the extreme-value theorem. The distance can be expressed as a function of *x*. Since *x* is the distance from A” to P and P can be anywhere in L from A” to B”, the value of *x* is in the closed interval [0, *c*]. Note also that P should not be at the left of A” in L nor be at the right of B” in L since the distance is to be minimized. It can be easily shown, through the use of coordinate points and the distance formula in analytic geometry,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$, that

d(A, A”) + d(A”, B) < d(A, P at left of A”) + d(P at left of A”, B) and

d(B, B”) + d(B”, A) < d(B, P at right of B”) + d(P at right of B”, A).

For $d = y + z$, the application of the Pythagorean Theorem results to

$$d = \sqrt{x^2 + a^2} + \sqrt{(c - x)^2 + b^2} \,,$$

the derivative of which is

$$\frac{d(d)}{dx} = \frac{x}{\sqrt{x^2 + a^2}} + \frac{x - c}{\sqrt{(c - x)^2 + b^2}} \,.$$

Notice also that since d is a sum of two radical functions with positive even index 2 and whose radicands are polynomials, it is continuous on the closed interval [0, c], which can be shown through the application of the theorem on the continuity of a composite function, theorem on the addition of two continuous functions, definitions of continuity, and theorems on limits. For instance, if $f(x) = \sqrt{x}$ and $g(x) = x^2 + a^2$, then $f(g(x)) = \sqrt{x^2 + a^2}$. Recall that $g(x)$ being a polynomial function $x^2 + a^2$ is continuous at every number while $f(x) = \sqrt{x}$ is continuous at every positive number, hence $\sqrt{x^2 + a^2}$ is continuous at $x^2 + a^2 > 0$. Then, using the definition of right-hand continuity and left-hand continuity it can be shown that is continuous on [0, c]. The same process can be done for the second radical function $\sqrt{(c - x)^2 + b^2}$. Finally, recall that the sum of two continuous functions is continuous.

Next is to find relative extremum by equating the derivative to zero, i.e.

$$\frac{x}{\sqrt{x^2 + a^2}} + \frac{x - c}{\sqrt{(c - x)^2 + b^2}} = 0 \,,$$

which, by series of algebraic manipulations, can be simplified into

$$(a^2 - b^2)x^2 - (2ca^2)x + c^2a^2 = 0 \,.$$

Then, by quadratic formula, the roots of the equation or the values of x are

$$x = \frac{ac}{a - b} \quad \text{and} \quad x = \frac{ac}{a + b} \,.$$

The first root, however, is an extraneous root and is therefore rejected. Recall that x should be in the closed interval [0, c]. In examining the root $\frac{ac}{a - b}$, there are three cases: (1) $a = b$, (2) $a < b$, and (3) $a > b$. If $a = b$, then obviously $\frac{ac}{a - b}$ is undetermined. If $a < b$, then $\frac{ac}{a - b} < 0$, which is not in [0, c]. If $a > b$, then there exists a positive number r such that $a = b + r$ or $r = a - b$ and hence $\frac{ac}{a - b} = \frac{(b + r)c}{r} = \frac{bc}{r} + c$, which is obviously greater than c and so, not in [0, c].

Hence, the relative extremum is $x = \frac{ac}{a+b}$. Looking back at Figure 8, this can be considered as the formula for determining the location of P in L from A”. Notice that if $a = b$, i.e. when A and B have equal distances from L, then $x = \frac{c}{2}$ or P is in the middle of A” and B”. It can be shown also that P is always nearer the point with shorter distance from L.

The distance *d* can now be expressed in terms of the distances *a*, *b* and *c*, i.e.

$$d = \sqrt{\left(\frac{ac}{a+b}\right)^2 + a^2} + \sqrt{\left(c - \frac{ac}{a+b}\right)^2 + b^2}$$

that when simplified will result to

$$d = \sqrt{(a+b)^2 + c^2} \ ,$$

and which is the distance from A to P in L to B at the relative extremum $x = \frac{ac}{a+b}$. With the extreme-value theorem in mind, the next step is to compute the distance at the endpoints of [0, *c*]. That is, if $x = 0$, then $d = a + \sqrt{b^2 + c^2}$ and if $x = c$, then $d = b + \sqrt{a^2 + c^2}$.

To determine if $d = \sqrt{(a+b)^2 + c^2}$ is the absolute minimum, there is a need to show that $\sqrt{(a+b)^2 + c^2}$ is less than both $a + \sqrt{b^2 + c^2}$ and $b + \sqrt{a^2 + c^2}$.

Hence, $\sqrt{(a+b)^2 + c^2} < a + \sqrt{b^2 + c^2}$, which when both sides are squared and the inequality is simplified results to $b < \sqrt{b^2 + c^2}$ and squaring again gives the obviously true inequality $b^2 < b^2 + c^2$. Similarly, $\sqrt{(a+b)^2 + c^2} < b + \sqrt{a^2 + c^2}$, which when also squared and simplified will have $a < \sqrt{a^2 + c^2}$ and squaring again gives also a true inequality $a^2 < a^2 + c^2$.

It is now very clear that $d = \sqrt{(a+b)^2 + c^2}$ is a formula that will surely give the minimum distance from a point to a line to another point on the same side of the line. The formula, being a Pythagorean relation and as shown in the geometrical derivation, is also applicable for the minimum distance from a point to a line to another point on the other side of the line.

3.3. Problems and Solutions

Sample problems and solutions to illustrate the use of the derived formulas are presented here. In solving the following problems, figures or illustrations may or may not be used, as long as the distances, *a*, *b* and *c*, are clear for the problem solver.

Problem 1. Towns A and B, situated inland on the same side of a straight river, want to erect a jointly owned pumping station P on the bank of the river. If town A is 12 km from the river, town B is 28 km from the river, and the points on the river nearest to A and B are 30 km apart, where should P be located if the length of the pipe to be used from P to A and from P to B is to be a minimum? What will be the minimum length of the pipe?

Solution: Using the derived formula for the relative extremum value of x, the location of P can be easily found with a = 12 km, b = 28 km and c = 30 km as follows:

$$x = \frac{ac}{a+b} = \frac{12(30)}{12+28} = 9 \cdot$$

For minimum length of pipe to be used, the derived minimum distance formula can be used, i.e.

$$d = \sqrt{(a+b)^2 + c^2} = \sqrt{(12+28)^2 + 30^2} = 50 \, .$$

Therefore, the location of P is 9 km from the point on the river nearest A and the minimum length of the pipe to be used is 50 km. The illustration of the above problem is given in Figure 9.

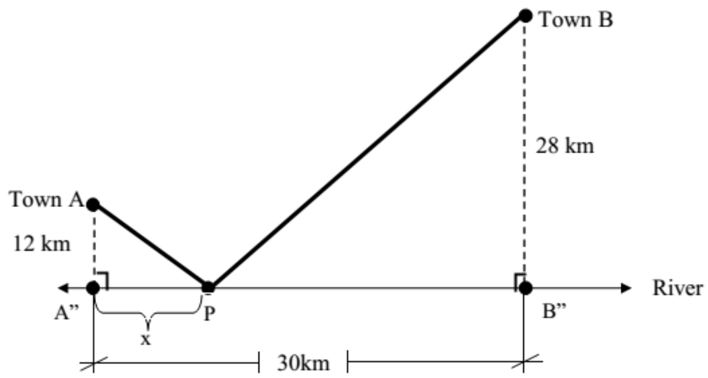

Figure 9: Problem 1 Illustration

Problem 2. Suppose that light reflects off a flat mirror to get from point A to point B. Assuming a constant velocity of light, time can be minimized by minimizing the distance traveled. If A is 2 units and B is 1 unit from the mirror and the points on the mirror nearest A and B are 4 units apart, find the location of point P on the mirror that minimizes the distance traveled.

Solution: Given that a = 2, b = 1 and c = 4, the location of P that minimizes the distance traveled by the light is $x = \frac{ac}{a+b} = \frac{2(4)}{2+1} \approx 2.667$ units from the point on the mirror nearest A.

Problem 3. Two posts, both 15 feet high, stand 24 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. What is the minimum total length of wire possible if the stake can be placed anywhere on the ground?

Solution: Since the two posts are both 15 ft high and 24 ft apart, the minimum length of wire possible is $d = \sqrt{(a+b)^2 + c^2} = \sqrt{(15+15)^2 + 24^2} \approx 38.419$ ft.

4. Conclusion and Directions for Future Works

Solving optimization problems is a very important application of differential calculus. This application is extended to actual business undertakings and engineering works, where most often, the main goal is maximizing profit and minimizing cost. Minimizing length or distance, and in a way, minimizing the cost of materials for projects involving lengths or distances, is an example of such application.

To solve problems of minimizing distance, the extreme-value theorem is usually applied. This involves finding the derivative of a function, finding critical points or relative extrema, and calculating the distance as a function value at the critical points and at the endpoints of the closed interval and then comparing the results to find the minimum. However, such solution is somewhat lengthy and prone to computational errors.

Thus, using a formula in solving minimum distance problems is a better alternative; the solution is direct and can be done with greater ease, accuracy and speed. In particular, the minimum distance from a point to a line to another point on the same side of the line can be determined by the formula $d = \sqrt{(a+b)^2 + c^2}$, where *d* is the minimum distance, *a* is the distance from one point to the line, *b* is the distance from the other point to the line, and *c* is the distance between the two points in the line nearest the two given points. This formula is derived using two different approaches: (1) geometrical construction and reasoning and (2) analytical application of extreme-value theorem. In the derivation process, the formula for finding the location of the point in the line that will lead to having minimum distance was also determined i.e. $x = ac/(a+b)$.

With the use of the derived formulas, the problem solver will no longer need to repeat the same reasoning process in geometry, or the same lengthy computations in differentiation, over and over again whenever there is a need to solve a minimum distance problem of this kind. In practical applications, if the minimum distance was accurately determined then waste of materials can be avoided and hence, minimizing cost. Also, finding the minimum distance directly through the use of formula will greatly reduce time and manpower requirement thereby reducing project cost.

Future works to be done by other research mathematicians may include (1) determining the minimum distance formula from a point to a line to another point to the same line to a third point and so on, i.e. minimum distance formula that involves a line and more than two points on one side or both sides of the line and (2) deriving formulas for other optimization applications of differential calculus. Practitioners in the fields of business and engineering may further develop the utilization of the derived formula or of future similar mathematical models in bigger construction and business projects for example, a new business center to be located in a highway from where road projects with least possible distances are to be built going to urban centers, an airfield to be developed in a tourism zone whereby also minimizing the distances to different tourism establishments, or a bullet train central station connecting various destinations in the shortest possible distance and time.

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